Lecture 1: Linear Algebra

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Linear algebra is the most fundamental pillar of linear systems and controls. A comprehensive coverage of linear algebra can take years!, and is way beyond our scope here. In this lecture I cover only some of the basic concepts and results that we will use later in the course. For a nice and more comprehensive treatment, but still without proofs, see Chapter 3 of the Chen's textbook. For a pretty comprehensive treatment, you can take a stop at Tom Bewley's encyclopedia, "Numerical Renaissance: simulation, optimization, & control". If you want just something in between, which is nicely readable and moderately comprehensive, try Gilbert Strang's "Introduction to Linear Algebra".

Contents

1.1   Vectors and Matrices

You probably know very well what a vector and a matrix are.

Definition 1.1.1 (Scalars, vectors, and matrices) A "scalar", for the purpose of this course, is either a real ($\mathbb{R}$) or a complex ($\mathbb{C}$) number. A "vector", again for the purpose of this course, is an ordered set of numbers, depicted as a column:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1}$$

Almost always, our vectors are column vectors. But occasionally, we need row vectors as well, which we may show using the transpose ^{T} notation:

$$\mathbf{x}^T = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}_{1 \times n}$$

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And finally a "matrix" is a rectangular ordered set of numbers,

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

As you have noticed, throughout this course, we use bold-faced small letters for vectors and bold-faced capital letters for matrices. □

Vectors and matrices of the same size can be added together, and both vectors and matrices can be multiplied by a scalar, not super interesting. What is more interesting and, as you will see, essentially the basis of linear algebra, is matrix multiplication. You probably have seen the basic definition.

Definition 1.1.2 (Matrix multiplication) For two matrices $\mathbf{A}_{m \times n}$ and $\mathbf{B}_{r \times p}$, their product $\mathbf{C} = \mathbf{AB}$ is only defined if $n = r$, in which case the entries of $\mathbf{C}_{m \times p}$ are defined as

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

□

The above definition gives little intuition about what a matrix multiplication really does. To see this, we need to notice two facts.

Theorem 1.1.3 (Breakdown of matrix multiplication) Let $\mathbf{a}_i$ and $\mathbf{b}_i$ denote the i'th column of $\mathbf{A}$ and $\mathbf{B}$, respectively:

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_p \end{bmatrix}$$

Then

(i) The <u>matrix-matrix multiplication</u> $\mathbf{AB}$ applies to each column of $\mathbf{B}$ separately, that is

$$\mathbf{A} \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_p \end{bmatrix} = \begin{bmatrix} \mathbf{Ab}_1 & \mathbf{Ab}_2 & \cdots & \mathbf{Ab}_p \end{bmatrix} \tag{1.1}$$

In other words, the i'th column of $\mathbf{AB}$ is $\mathbf{A}$ times the i'th column of $\mathbf{B}$.

(ii) Each <u>matrix-vector multiplication</u> $\mathbf{Ab}_i$ is a weighted sum of the columns of $\mathbf{A}$, that is

$$\mathbf{Ab}_i = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} b_{1i} \\ b_{2i} \\ \vdots \\ b_{ni} \end{bmatrix} = b_{1i} \mathbf{a}_1 + b_{2i} \mathbf{a}_2 + \cdots + b_{ni} \mathbf{a}_n \tag{1.2}$$

□

You can easily show both of these properties using Definition 1.1.2, but they are going to be super useful in understanding linear algebra as it really is.

Exercise 1.1.4 (Breakdown of matrix multiplication) Use the above two rules to compute $\mathbf{AB}$, where

$$\mathbf{A} = \begin{bmatrix} 0 & 2 & 1 \\ -1 & 1 & -3 \\ 3 & -1 & 3 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} -4 & 0 \\ 2 & 1 \\ -1 & 0 \end{bmatrix}$$

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□

You might be wondering why I put so much emphasis on these simple properties? One reason is the intuition that they give you on matrix multiplication. But it's more than that. The real reason is the important role that matrix-vector multiplication plays in linear algebra. Notice that for a matrix $\mathbf{A}_{m \times n}$ and a vector $\mathbf{x} n \times 1$, their product

$$\mathbf{y}_{m \times 1} = \mathbf{A}\mathbf{x}$$

is also a vector. In other words, a matrix is more than a rectangular array of numbers! It defines a function that maps vectors to vectors.
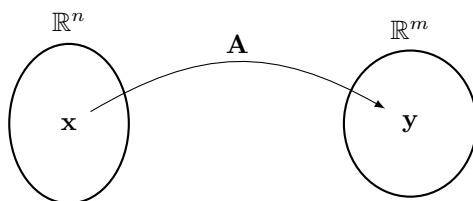
This view of matrices as maps (or operators) is arguably the basis of linear algebra!

1.2 Linear (In)dependence

Matrices, when viewed as maps, have an important property: they are <u>linear</u>:

$$\mathbf{A}(\alpha_1 \mathbf{x}_1 + \alpha_2 \mathbf{x}_2) = \alpha_1 \mathbf{A}\mathbf{x}_1 + \alpha_2 \mathbf{A}\mathbf{x}_2 \tag{1.3}$$

where $\mathbf{x}_1, \mathbf{x}_2$ are vectors and α_1, α_2 are scalars. In other words, if you have already computed $\mathbf{y}_1 = \mathbf{A}\mathbf{x}_1$ and $\mathbf{y}_2 = \mathbf{A}\mathbf{x}_2$, and you want to compute $\mathbf{A}(\alpha_1 \mathbf{x}_1 + \alpha_2 \mathbf{x}_2)$, you don't have to use matrix multiplication (which is computationally expensive) anymore. You can simply reuse $\mathbf{y}_1$ and $\mathbf{y}_2$ and "combine" them using the same coefficients α_1 and α_2.

MATLAB 1.2.1 (Linearity of matrix multiplication) To see the advantage of using the right hand side of Eq. (1.3) over its left hand side, compare

```
1   n = 1e4;
2   x1 = rand(n, 1);
3   x2 = rand(n, 1);
4   A = rand(n);
5   tic
6   for i = 1:1e3
7       alpha1 = rand;
8       alpha2 = rand;
9       y = A * (alpha1 * x1 + alpha2 * x2);
10  end
11  toc
```

with

```
1   n = 1e4;
2   x1 = rand(n, 1);
```

4

```
3   x2 = rand(n, 1);
4   A = rand(n);
5   tic
6   y1 = A * x1;
7   y2 = A * x2;
8   for i = 1:1e3
9       alpha1 = rand;
10      alpha2 = rand;
11      y = alpha1 * y1 + alpha2 * y2;
12  end
13  toc
```

If you haven't seen them before, `rand(n, m)` generates a matrix of size $n \times m$ with random entries in the interval $[0, 1]$, `rand(n)` is equivalent to `rand(n, n)` (<u>not</u> `rand(n, 1)`), and `rand` is equivalent to `rand(1)`. The `tic, toc` pair compute the CPU time for all the commands computed in between. □

In general, terms of the form $\alpha_1 \mathbf{x}_1 + \alpha_2 \mathbf{x}_2$ appear over and over again in linear algebra, so they have been given a name!

Definition 1.2.2 (Linear combination) A linear combination of a set of vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ is any weighted sum of them, that is, any vector

$$\mathbf{x}_{n+1} = \alpha_1 \mathbf{x}_1 + \alpha_2 \mathbf{x}_2 + \cdots + \alpha_n \mathbf{x}_n \tag{1.4}$$

for some scalars (also called "coefficients") $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$. □

For obvious reasons, the vector $\mathbf{x}_{n+1}$ in Eq. (1.4) is said to be *linearly dependent* on the vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ (because you can obtain $\mathbf{x}_{k+1}$ from a linear combination of $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$). Notice that this is not always the case. For example, you cannot find any linear combination of

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

that gives you

$$\mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and, so, $\mathbf{x}_3$ is *linearly independent* from $\mathbf{x}_1$ and $\mathbf{x}_2$. But

$$\mathbf{x}_4 = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$$

is in fact linearly dependent on $\mathbf{x}_1$ and $\mathbf{x}_2$ (right?).

Here, you might have noticed that not only $\mathbf{x}_3$ cannot be obtained from any linear combination of $\mathbf{x}_1$ and $\mathbf{x}_2$, but $\mathbf{x}_2$ cannot be obtained from any linear combination of $\mathbf{x}_1$ and $\mathbf{x}_3$ either, and same for $\mathbf{x}_1$. On the other hand, not only $\mathbf{x}_4$ can be obtained from $\mathbf{x}_1$ and $\mathbf{x}_2$

$$\mathbf{x}_4 = 2\mathbf{x}_1 + 3\mathbf{x}_2$$

but also $\mathbf{x}_1$ can be obtained from $\mathbf{x}_4$ and $\mathbf{x}_2$

$$\mathbf{x}_1 = \frac{1}{2}\mathbf{x}_4 - \frac{3}{2}\mathbf{x}_2$$

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and $\mathbf{x}_2$ can be obtained from $\mathbf{x}_4$ and $\mathbf{x}_1$ as well

$$\mathbf{x}_2 = \frac{1}{3}\mathbf{x}_4 - \frac{2}{3}\mathbf{x}_1$$

In other words, (assuming that no coefficients are 0) linear dependence and linear independence are symmetric properties *between* a set of vectors: they are either all linearly dependent on each other, or neither is linearly dependent on the rest. The formal version is:

Definition 1.2.3 (Linear (in)dependence) A set of vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are "linearly dependent" if there exists a set of scalars $\alpha_1, \ldots, \alpha_n$, at least one of which is not equal to 0, such that

$$\alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + \cdots + \alpha_n\mathbf{x}_n = 0 \tag{1.5}$$

In contrast, if Eq. (1.5) holds only for $\alpha_1 = \cdots = \alpha_n = 0$ (which it clearly always does), then $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are called "linearly independent". $\qquad\square$

Now, if I give you a set of vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$, how can you say if they are linearly independent or not? For example,

$$\mathbf{x}_1 = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \qquad \mathbf{x}_2 = \begin{bmatrix} -3 \\ 3 \\ 1 \end{bmatrix}, \qquad \mathbf{x}_3 = \begin{bmatrix} 7 \\ -3 \\ 3 \end{bmatrix} \tag{1.6a}$$

are linearly independent, but

$$\mathbf{x}_1 = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \qquad \mathbf{x}_2 = \begin{bmatrix} -3 \\ 3 \\ 1 \end{bmatrix}, \qquad \mathbf{x}_3 = \begin{bmatrix} 7 \\ -3 \\ -3 \end{bmatrix} \tag{1.6b}$$

are not. There are various ways for determining it, and we will get back to it in more detail in Section 1.3. But for now, we can say the following.

Remark 1.2.4 (Determining linear independence) For now,

- First, notice that for two vectors, if they are linearly dependent, it means that

$$\alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 = 0$$

for some α_1 and α_2 that at least one of them, say α_1, is nonzero. This means that

$$\mathbf{x}_2 = -\frac{\alpha_2}{\alpha_1}\mathbf{x}_1$$

or, in words, two vectors are linearly dependent if and only if they are a multiple of each other.

- Second, if you have n, m-dimensional vectors and $n > m$, they are necessarily linearly dependent. So in $\mathbb{R}^2$, you cannot have 3 linearly independent vectors, in $\mathbb{R}^3$, you cannot have 4 linearly independent vectors, and so on. We will see why shortly, in Section 1.3.

- Also, just remember that you can always check linear independence directly from Definition 1.2.3. At the end of the day, Eq. (1.5) is a system of linear equations with unknowns $\alpha_1, \alpha_2, \ldots, \alpha_n$. Later in this note I will discuss systems of linear equations in detail, but you can always solve them manually (for example using the substitution method, reviewed *here*). If the only answer is $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$, then the vectors are linearly independent. If you fail to find a unique answer, then the vectors are linearly dependent.

□

Exercise 1.2.5 (Linear (in)dependence) Check whether any of the following set of vectors are linearly dependent or independent:

- Vectors in Eq. (1.6a)

- Vectors in Eq. (1.6b)

- The columns of

$$\mathbf{A} = \begin{bmatrix} 2 & 3 & -2 & 3 & -3 \\ 3 & 1 & 0 & -2 & 2 \\ -3 & -3 & 3 & 3 & -3 \end{bmatrix} \tag{1.7}$$

- The rows of $\mathbf{A}$ (or the columns of $\mathbf{A}^T$).

□

Determining linear independence by solving the linear system of equations is pretty tedious. In fact, we will later do the opposite (solve linear system of equations using linear independence). To determine linear independence more easily, we need rank and determinant.

1.3 Rank and Determinant

For a set of vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n \in \mathbb{R}^m$, we start by placing them into a matrix (as columns):

$$\mathbf{A} = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \end{bmatrix}$$

So the matrix $\mathbf{A}$ has n columns and m rows. Notice that this same matrix can also be seen as a stack of m, n-dimensional row vectors

$$\mathbf{A} = \begin{bmatrix} \mathbf{y}_1^T \\ \mathbf{y}_2^T \\ \vdots \\ \mathbf{y}_m^T \end{bmatrix}$$

where each <u>row vector</u> $\mathbf{y}_i^T$ denotes the i'th row of $\mathbf{A}$ (or, equivalently, the column vector $\mathbf{y}_i$ is the transpose of the i'th row of $\mathbf{A}$). Now, we are ready to define what the rank of a matrix is:

Definition 1.3.1 (Rank) Consider a matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ as above.

- The column-rank of $\mathbf{A}$ is the number linearly independent columns of $\mathbf{A}$ (the number of linearly independent $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$).

- The row-rank of $\mathbf{A}$ is the number of linearly independent rows of it (the number of linearly independent $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_m$).

□

The notion of rank essentially translate a property of a set of vectors to a property of a matrix, but is very fundamental to linear algebra. Let's see a couple of examples.

Example 1.3.2 (Rank of matrices) Consider again the example vectors in Eq. (1.6). The first set of vectors were linearly independent (as you checked in Exercise 1.2.5). Therefore, when we put them side by side in the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & -3 & 7 \\ 0 & 3 & -3 \\ -1 & 1 & 3 \end{bmatrix}$$

it has column rank equal to 3. What about its row rank? The rows of the matrix are

$$\mathbf{y}_1^T = \begin{bmatrix} 2 & -3 & 7 \end{bmatrix}, \qquad \mathbf{y}_2^T = \begin{bmatrix} 0 & 3 & -3 \end{bmatrix}, \qquad \mathbf{y}_3^T = \begin{bmatrix} -1 & 1 & 3 \end{bmatrix}$$

which are also linearly independent (check). So the row rank of the matrix is also 3.

Now, consider the vectors in Eq. (1.6b). They are not linearly independent, because $\mathbf{x}_3 = 2\mathbf{x}_1 - \mathbf{x}_2$. But $\mathbf{x}_1$ and $\mathbf{x}_2$ are indeed linearly independent, because they are not a multiple of each other (remember from last section). So putting them side by side, the matrix

$$\mathbf{B} = \begin{bmatrix} 2 & -3 & 7 \\ 0 & 3 & -3 \\ -1 & 1 & -3 \end{bmatrix}$$

has column rank 2. What about its row rank? The rows of $\mathbf{B}$ are

$$\mathbf{y}_1^T = \begin{bmatrix} 2 & -3 & 7 \end{bmatrix}, \qquad \mathbf{y}_2^T = \begin{bmatrix} 0 & 3 & -3 \end{bmatrix}, \qquad \mathbf{y}_3^T = \begin{bmatrix} -1 & 1 & -3 \end{bmatrix}$$

which are also not linearly independent, because $\mathbf{y}_3 = -\frac{1}{2}\mathbf{y}_1 - \frac{1}{6}\mathbf{y}_2$. And similar to $\mathbf{x}_1$ and $\mathbf{x}_2$, $\mathbf{y}_1$ and $\mathbf{y}_2$ are also linearly independent because they are not a multiple of each other. So, exactly 2 of $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3$ are linearly independent, and the row rank of $\mathbf{B}$ is also 2. $\qquad\square$

The fact that the column rank and the row rank of both $\mathbf{A}$ and $\mathbf{B}$ were equal was not a coincident. This is always the case:

Theorem 1.3.3 (Rank) For any matrix $\mathbf{A}_{m\times n}$, its column rank equals its row rank, which is called the rank of the matrix. As a consequence,

$$\text{rank}(\mathbf{A}) \le \min\{m, n\}. \tag{1.8}$$

$\qquad\square$

Recall that in the previous section (second point after Definition 1.2.3), I told you that if you have n, m-dimensional vectors and $n > m$, they are necessarily linearly dependent. Now you can see why from Eq. (1.8). For example, consider the same $\mathbf{A}$ matrix in Eq. (1.7). The rank of the matrix can at most be 3, because it has only 3 rows and its row-rank (number of independent rows) cannot be more than the number of rows! In this case, the rank is in fact 3, which means that out of the 5 columns, no more than 3 of them can be simultaneously independent form each other.

MATLAB 1.3.4 (Rank) Compute the rank of a matrix using `rank`. $\qquad\square$

Now you know how to easily check the independence of a set of vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$: stack them side by side in a matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{x}_1 & \cdots & \mathbf{x}_n \end{bmatrix}$$

and check the rank of $\mathbf{A}$ (using MATLAB for example). If

$$\text{rank}(\mathbf{A}) = \#columns = n$$

the vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are linearly independent, and dependent if $\text{rank}(\mathbf{A}) < n$. This motivates the following definition:

Definition 1.3.5 (Full rank matrices) A matrix $\mathbf{A}_{m \times n}$ is

- full column rank if $\text{rank}(\mathbf{A}) = n$;
- full row rank if $\text{rank}(\mathbf{A}) = m$;
- full rank if $\text{rank}(\mathbf{A}) = \min\{m, n\}$.

□

However, the problem of determining linear independence doesn't end here. Consider the same $\mathbf{A}$ in Eq. (1.7). $\text{rank}(\mathbf{A}) = 3 < 5$, so the columns of $\mathbf{A}$ are not linearly independent. You know even more, that among the 5 columns, at most 3 are linearly independent. But which 3? **Note**, that column rank $= 3$ does not mean that <u>any</u> selection of 3 columns are linearly independent. Clearly, the last column is minus the 4th column, so any selection of 3 columns that include both the fourth and last columns will be linearly dependent. Instead, rank $= 3$ means that <u>at least one</u> selection of 3 columns are linearly independent. For example, the first, second, and third columns are linearly independent, and so are the first, second, and fourth columns.

A notion closely related to rank and linear (in)dependence is determinant.

Definition 1.3.6 (Determinant) Consider any n-by-n matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$.

- If $n = 1$ (a is a scalar), then its determinant equals itself

$$\det(a) = a$$

- If $n = 2$,

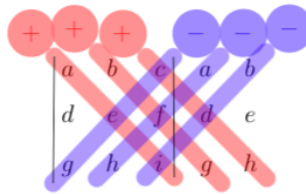$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\det(\mathbf{A}) = ad - bc$$

- If $n = 3$,

$$\mathbf{A} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$\det(\mathbf{A}) = aei + bfg + cdh - ceg - bdi - afh$$

Which is much easier to remember and calculate using the following image:

- If $n > 1$ (including $n = 2$ and 3 above, but really used for $n \geq 4$), then the determinant of $\mathbf{A}$ is defined based on an expansion over any arbitrary row or column. For instance, choose row 1. Then
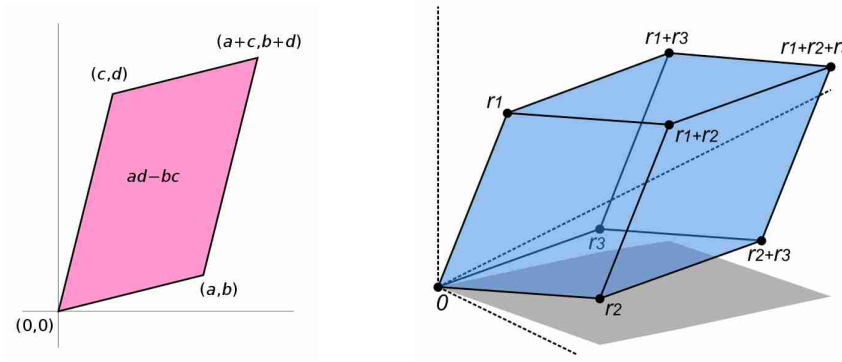
$$\det(\mathbf{A}) = |\mathbf{A}| = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(\mathbf{A}_{-(1,j)})$$

where $\mathbf{A}_{-(1,j)}$ is an $n - 1$ by $n - 1$ matrix obtained from $\mathbf{A}$ by removing its 1st row and j'th column.

We sometimes use $|\mathbf{A}|$ instead of $\det(\mathbf{A})$. ☐

The determinant also has a nice geometrical interpretation,

when $n = 2$: $|\det(\mathbf{A})| =$ area of the parallelogram formed by the columns (or rows) of $\mathbf{A}$

when $n = 3$: $|\det(\mathbf{A})| =$ volume of the parallelepiped formed by the columns (or rows) of $\mathbf{A}$

The main use of determinants for us, however, is not their geometrical interpretation, but rather their relation to independence of vectors (finally!):

Theorem 1.3.7 (Rank and determinant) Consider a matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$.

(i) If $m = n$ (square matrix), then

$$\det(\mathbf{A}) \neq 0 \Leftrightarrow \operatorname{rank}(\mathbf{A}) = n \Leftrightarrow \text{ all columns/rows of } \mathbf{A} \text{ are linearly independent}$$

(ii) In general (square matrix or not),

$$\operatorname{rank}(\mathbf{A}) = \text{ dimension of largest square sub-matrix that is nonsingular}$$

A matrix is called "nonsingular" if its determinant is nonzero, and "singular" otherwise. ☐

To see how to apply this theorem, let us revisit Example 1.3.2. First,

$$\det(\mathbf{A}) = 36 \neq 0$$

so the rank of $\mathbf{A}$ is 3 and all of its columns/rows are independent. For $\mathbf{B}$, we have

$$\det(\mathbf{B}) = 0$$

and so the rank of $\mathbf{B}$ cannot be 3. But still, its rank may be 2, 1, or even 0 (which is clearly not the case, because the rank of a matrix is only 0 if all of its entries are 0). To see if its rank is 2 or 1, we have to find the largest square sub-matrix that is nonsingular. Easily, you can find many 2×2 nonsingular submatrices, for example

$$\begin{bmatrix} 2 & -3 \\ 0 & 3 \end{bmatrix}, \quad \begin{bmatrix} 2 & -3 \\ -1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 2 & 7 \\ -1 & -3 \end{bmatrix}$$

so the rank of $\mathbf{B}$ is 2. To also see an example of a matrix with rank 1, take

$$\mathbf{C} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$$

All columns (or rows) are multiples of each other, and you cannot find any 2×2 nonsingular submatrices. You can, however, find 1×1 nonsingular submatrices (any entry of $\mathbf{C}$), which means $\operatorname{rank}(\mathbf{C}) = 1$.

MATLAB 1.3.8 (Determinant) In MATLAB, use the function `det` to obtain the determinant of a matrix.
□

Before finishing this section, let's see a couple of basic properties of the determinant:

Theorem 1.3.9 (Determinant of product and transpose) For two square matrices $\mathbf{A}$ and $\mathbf{B}$,

$$\det(\mathbf{A}^T) = \det(\mathbf{A})$$

and

$$\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B}).$$

□

Exercise 1.3.10 (Rank using definition and determinant) Compute that rank of

$$\mathbf{A} = \begin{bmatrix} -3 & -3 & -4 & -4 \\ 3 & 3 & 4 & 4 \\ 0 & 1 & -4 & 2 \end{bmatrix}$$

in 3 ways: from Definition 1.2.3, from Theorem 1.3.7, and using MATLAB. From both of the first two methods, determine a maximal set of linearly independent columns. □

1.4   Linear Systems of Equations

The notions of rank and determinant not only help with determining whether a set of vectors are independent or not, they also help with solving linear systems of equations.

Consider a general linear system of equation

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_1$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

containing m equations and n unknowns $x_1, \ldots, x_n$. Now you can easily see that this system can be written in matrix form as

$$\mathbf{Ax} = \mathbf{b} \tag{1.9}$$

So the question is: for a given $\mathbf{A}$ and $\mathbf{b}$, find <u>all</u> $\mathbf{x}$ that solve this equation. In general, Eq. (1.9) can have 0, 1, or infinitely many solutions.

The above question is essentially composed of two questions:

1) Does there exist any solutions to Eq. (1.9)?

2) If any solutions exist, is it unique?

We answer them in order:

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1.4.1 Existence of solutions

Recall from the end of Section 1.1 that $\mathbf{A}$ defines a map from $\mathbb{R}^n$ to $\mathbb{R}^m$. Now, we are given a specific vector $\mathbf{b}$ in $\mathbb{R}^m$ and asked whether there are any vectors $\mathbf{x}$ in $\mathbb{R}^n$ that map to it. Again, the property in Eq. (1.2) comes in handy. Notice that

$$\mathbf{A}\mathbf{x} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \mathbf{a}_1 + x_2 \mathbf{a}_2 + \cdots + x_n \mathbf{a}_n$$

so whether Eq. (1.9) has a solution is the same as asking whether $\mathbf{b}$ is linearly dependent on the columns of $\mathbf{A}$. To check this, you can just create a larger augmented matrix

$$\mathbf{A}_{\text{aug}} = \begin{bmatrix} \mathbf{A} & \mathbf{b} \end{bmatrix}$$

where you append $\mathbf{b}$ to the columns of $\mathbf{A}$. Then

if $\text{rank}(\mathbf{A}_{\text{aug}}) = \text{rank}(\mathbf{A}) \Rightarrow \mathbf{b}$ is linearly dependent on columns of $\mathbf{A} \Rightarrow$ at least 1 solution exists

if $\text{rank}(\mathbf{A}_{\text{aug}}) > \text{rank}(\mathbf{A}) \Rightarrow \mathbf{b}$ is linearly independent from columns of $\mathbf{A} \Rightarrow$ no solutions exist

(what if $\text{rank}(\mathbf{A}_{\text{aug}}) < \text{rank}(\mathbf{A})$?) and you can check the ranks either using determinants via Theorem 1.3.7, or directly using `rank()` in MATLAB. Note that if you are using the determinants, your job often becomes easier if you remove any linearly dependent columns of $\mathbf{A}$ before appending $\mathbf{b}$ to it (right?).

1.4.2 Uniqueness of Solutions

Let us continue with the example above.

Example 1.4.3 (Uniqueness of solutions) Consider the same $\mathbf{A}$ and $\mathbf{b}$ as above,

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 5 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

I gave one solution $\mathbf{x}$ above, but that is not the only solution. $\mathbf{x}' = \begin{bmatrix} 2 & 0 & -1 \end{bmatrix}^T$ also solves this equation, so does $\mathbf{x}'' = \begin{bmatrix} 3 & 1 & -2 \end{bmatrix}^T$, and many others. Why is this? Because

$$\mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3 = 0 \Rightarrow \mathbf{A} \underbrace{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}}_{\mathbf{z}} = 0$$

And so, for any scalar α,

$$\mathbf{A}(\mathbf{x} + \alpha \mathbf{z}) = \mathbf{A}\mathbf{x} + \alpha \mathbf{A}\mathbf{z} = \mathbf{b} + 0 = \mathbf{b}$$

so all $\mathbf{x} + \alpha \mathbf{z}$ are solutions as well. Note that this only happens because there exists a <u>nonzero</u> vector $\mathbf{z}$ such that $\mathbf{A}\mathbf{z} = 0$. □

The above examples motivates the definition of another fundamental concept in linear algebra:

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Definition 1.4.4 (Null space) The null space of a matrix $\mathbf{A}$ is the set

$$\{\mathbf{z} \mid \mathbf{A}\mathbf{z} = 0\}.$$

□

Let me emphasize that the null space is never empty because it always contains at least the zero vector.

So to determine whether the solution to $\mathbf{A}\mathbf{x} = \mathbf{b}$ (assuming that at least one exists) is unique, we need to determine whether the null space of $\mathbf{A}$ contains any nonzero vectors. Again, Eq. (1.2) comes in handy!

$$\mathbf{A}\mathbf{z} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} = z_1\mathbf{a}_1 + z_2\mathbf{a}_2 + \cdots + z_n\mathbf{a}_n$$

and so the question of whether

$$\mathbf{A}\mathbf{z} = 0$$

for a nonzero $\mathbf{z}$ is precisely the question of whether

$$z_1\mathbf{a}_1 + z_2\mathbf{a}_2 + \cdots + z_n\mathbf{a}_n = 0$$

for a nonzero set of coefficients $z_1, z_2, \ldots, z_n$, which is precisely the definition of linear independence! So,

$$\underline{\text{The solution to } \mathbf{A}\mathbf{x} = \mathbf{b} \text{ (if any) is unique} \Leftrightarrow \mathbf{A} \text{ has full column rank}} \tag{1.10}$$

When the solution to the equation $\mathbf{A}\mathbf{x} = \mathbf{b}$ is not unique, then you essentially have a *solution space* rather than a solution. Obtaining the solution space is very easy once you have the null space: if $\{\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_k\}$ is a basis for the null space of $\mathbf{A}$ and $\mathbf{x}$ is one solution to the equation, then

$$\text{Solution space} = \{\mathbf{x} + \alpha_1\mathbf{z}_1 + \alpha_2\mathbf{z}_2 + \cdots + \alpha_k\mathbf{z}_k \mid \alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{R}\}. \tag{1.11}$$

1.4.5 Finding the Solutions

At this point you might ask: ok, even if a unique solution exists, how to find it? This is done using the notion of matrix inverse:

Definition 1.4.6 (Matrix inverse) For a <u>square and nonsignular</u> matrix $\mathbf{A}$, there exists a unique matrix $\mathbf{A}^{-1}$ such that

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

where $\mathbf{I}$ is the identity matrix

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}.$$

□

You may remember that for a 2 by 2 matrix, its inverse is given by

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

For larger matrices, we will use MATLAB to find their inverses.

Let us now see how we can use matrix inverse to solve systems of linear equations:

- If $m = n$ ($\mathbf{A}$ is square) and $\mathbf{A}$ is nonsingular, then any $\mathbf{b}$ is linearly dependent on the columns of $\mathbf{A}$. Therefore, at least one solution exists. But this solution is also unique from Eq. (1.10), because $\mathbf{A}$ has full column rank as well. To find this unique solution, simply multiply both sides of $\mathbf{Ax} = \mathbf{b}$ by $\mathbf{A}^{-1}$ *from the left*:

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} \tag{1.12}$$

- If $m > n$ ($\mathbf{A}$ is a tall matrix), and $\mathbf{A}$ is full column rank (it cannot be full row-rank, right?), then we have a unique solution only if $\mathbf{b}$ is linearly dependent on the columns of $\mathbf{A}$. *If so*, to find that solution,

$$\mathbf{Ax} = \mathbf{b} \Rightarrow \mathbf{A}^T\mathbf{Ax} = \mathbf{A}^T\mathbf{b} \Rightarrow (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{Ax} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{b} \Rightarrow \mathbf{x} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{b}$$

- If $m < n$ ($\mathbf{A}$ is a fat matrix), the equation never has a unique solution (even if $\mathbf{A}$ is full row-rank), because $\mathbf{A}$ cannot be full column rank (right?). Given any solution (that you can find, e.g., using elimination of variables), you can build the entire solution space as in Eq. (1.11).

  If, however, $\mathbf{A}$ is full row rank, then we are sure that the system of equations has always a solution for any $\mathbf{b}$ (why?). In this case, we also know (from a theorem we don't prove) that the square matrix $\mathbf{AA}^T$ is nonsingular, and

$$\mathbf{x} = \mathbf{A}^T(\mathbf{AA}^T)^{-1}\mathbf{b}$$

  is one solution to the equation (just plug it in and check!).

If you compare the three case above, you will notice that the matrices $(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T$ (for full column rank $\mathbf{A}$) and $\mathbf{A}^T(\mathbf{AA}^T)^{-1}$ (for full row rank $\mathbf{A}$) are essentially taking the place of $\mathbf{A}^{-1}$ in Eq. (1.12). In fact, if $\mathbf{A}$ is square and nonsingular, both of them will become equal to $\mathbf{A}^{-1}$ (because $(\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T$ and $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$). In other words, $(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T$ is an extension of $\mathbf{A}^{-1}$ for non-square, full column rank matrices and $\mathbf{A}^T(\mathbf{AA}^T)^{-1}$ is an extension of $\mathbf{A}^{-1}$ for non-square, full row rank matrices. As such, both of them are called the "pseudo-inverse" of $\mathbf{A}$, shown as $\mathbf{A}^\dagger$. Therefore, whenever $\mathbf{A}$ has full rank,

$$\mathbf{x} = \mathbf{A}^\dagger\mathbf{b} \tag{1.13}$$

is a solution to $\mathbf{Ax} = \mathbf{b}$. It goes beyond our course, but be aware that $\mathbf{A}^\dagger$ is always defined for any matrix (even the zero matrix) and Eq. (1.13) is always a solution to $\mathbf{Ax} = \mathbf{b}$.

MATLAB 1.4.7 (Inverse and pseudo-inverse) To find the inverse of a matrix, use the `inv()` function. Similarly, use `pinv()` to find the pseudo-inverse. However, if you want to invert a matrix only for the purpose of solving a linear equation, as in Eq. (1.12) or Eq. (1.13), a computationally advantageous way is to use the MATLAB's left division:

```
1  x = inv(A) * b; % Using matrix inverse
2  x = pinv(A) * b; % Using matrix pseudo-inverse
3  x = A \ b; % Using left division
```

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Using left division also allows you to solve systems of equations without a unique solution, or even non-square systems of equations. If your system of equations has infinitely many solutions, `A \ b` returns one of them. If your system has no solutions, then it returns an $\mathbf{x}$ for which the error $\mathbf{Ax} - \mathbf{b}$ is smallest (in magnitude). $\square$

1.5   Eigenvalues and Eigenvectors

OK, you have made it so far, and we are finally ready to learn about eigenvectors and eigenvalues! You'll see later why I put so much emphasis on them – they are one of the most important and widely used constructs in control theory.

Throughout this section, I will focus on square matrices $\mathbf{A} \in \mathbb{R}^{n \times n}$, because eigenvalues and eigenvectors are only defined for square matrices.

Recall, from Section 1.1, that matrices are not just arrays of numbers, but mappings from one vector space to another. So $\mathbf{A}$ maps from $\mathbb{R}^n$ to $\mathbb{R}^n$. In some cases, we can very easily describe what this mapping does:

Example 1.5.1 (2D mappings) Consider a few simple mappings in two dimensions:

- $\mathbf{A} = \mathbf{I}$ maps any vector to itself (identity mapping)

- $\mathbf{A} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ reflects any vector with respect to the vertical axis

$$\mathbf{A} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}$$

  Similarly, $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ reflects any vector with respect to the horizontal axis.

- $\mathbf{A} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} = k\mathbf{I}$ for $k > 0$ scales any vector by a factor of k

$$\mathbf{A} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} kx \\ ky \end{bmatrix}$$

- $\mathbf{A} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -\mathbf{I}$ reflects any vector with respect to the origin

$$\mathbf{A} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix}$$

- $\mathbf{A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ rotates any vector as much as θ radians counter-clockwise.

$\square$

What about more complex matrices? For example, how can we describe (or even intuitively understand) what does

$$\mathbf{A} = \begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix} \tag{1.14}$$

do to vectors? Here is how it maps a whole bunch of random points (the red dot shows the origin, the blue dots are random $\mathbf{x}$'s, the red dots are the corresponding $\mathbf{Ax}$, and the arrow shows the mapping):

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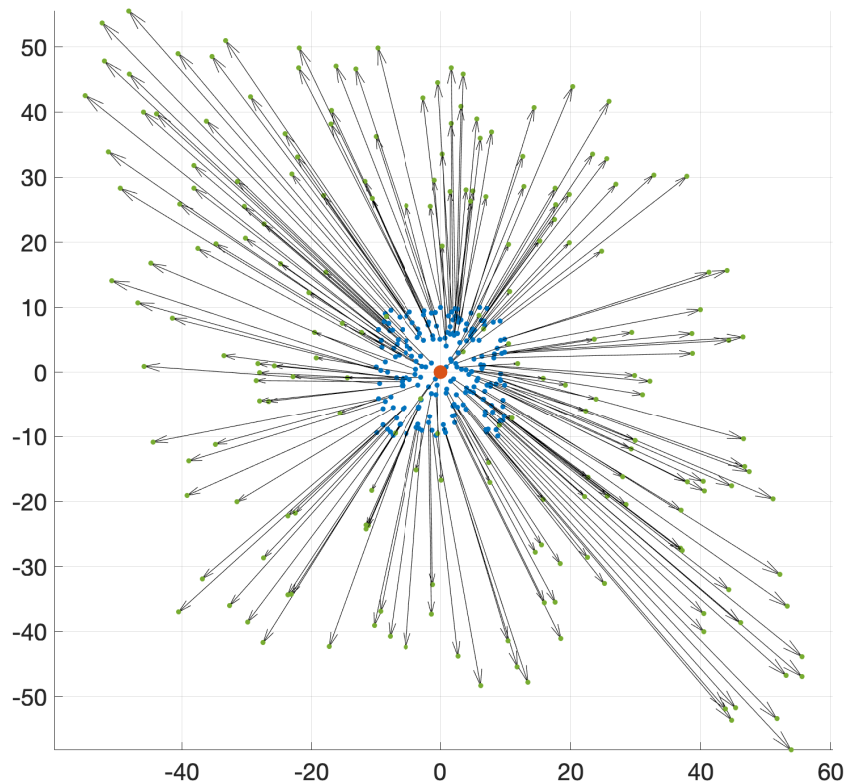
It already gives us a sense: The arrows are all pointing outwards, showing that $\mathbf{A}$ perform some sort of enlargement (scaling with a scale $k > 1$). But this enlargement is not uniform, it is more pronounced along a NorthWest-SouthEast axis. Notice that this NorthWest-SouthEast axis can be described by the vector

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

This vector is indeed special, since

$$\mathbf{A}\mathbf{v}_1 = \begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ 6 \end{bmatrix} = 6\mathbf{v}_1$$

In other words, $\mathbf{v}_1$ is special because when $\mathbf{A}$ acts on it, the result is a multiple of $\mathbf{v}_1$ again! In other words, the effect of $\mathbf{A}$ on $\mathbf{v}_1$ is a pure scaling. (If you think this is not so special, try a whole bunch of random vectors and check if $\mathbf{A}\mathbf{v}$ becomes exactly a multiple of $\mathbf{v}$.)

Note that this special property also clearly holds for any multiple of $\mathbf{v}_1$:

$$\mathbf{A}(\alpha\mathbf{v}_1) = \alpha\mathbf{A}\mathbf{v}_1 = \alpha 6\mathbf{v}_1 = 6(\alpha\mathbf{v}_1)$$

In other words, the effect of $\mathbf{A}$ on the whole NorthWest-SouthEast axis is a 6-time enlargement.

But this does not tell us all about $\mathbf{A}$. What about other directions other than the NorthWest-SouthEast axis? We can visually see that no other direction is scaled as much. To see what $\mathbf{A}$ does to other vectors, we

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can search to see if there are any other vectors such that the effect of $\mathbf{A}$ on them is a pure scaling. In other words, are there any vectors $\mathbf{v}$, other than $\mathbf{v}_1$ and its multiples, such that

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v} \tag{1.15}$$

for some scalar λ. Clearly, $\mathbf{v} = 0$ satisfies this, but that is not what we are looking for.

Eq. (1.15) is fortunately a linear system of equations:

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = 0$$

and we are looking for <u>nonzero</u> vectors $\mathbf{v}$ that satisfy it. The difficulty compared to a usual linear system of equations is that λ is also unknown. But notice one thing. If λ is such that $\mathbf{A} - \lambda\mathbf{I}$ is nonsingular, then

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = 0 \Rightarrow (\mathbf{A} - \lambda\mathbf{I})^{-1}(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = (\mathbf{A} - \lambda\mathbf{I})^{-1}0 \Rightarrow \mathbf{v} = 0$$

In other words, for any λ such that $\mathbf{A} - \lambda\mathbf{I}$ is nonsingular, Eq. (1.15) has only the trivial solution $\mathbf{v} = 0$, which is of no help. This is very useful, because we know we have to restrict our attention to values of λ for which $\mathbf{A} - \lambda\mathbf{I}$ is singular:

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0 \Leftrightarrow \det\left(\begin{bmatrix} 5 - \lambda & -1 \\ -1 & 5 - \lambda \end{bmatrix}\right) = 0$$
$$\Leftrightarrow (5 - \lambda)^2 - 1 = 0$$
$$\Leftrightarrow \lambda^2 - 10\lambda + 24 = 0$$

This already gives a polynomial equation in λ, with solutions

$$\lambda_1 = 6$$
$$\lambda_2 = 4$$

$\lambda_1 = 6$ is what we had originally found by guessing $\mathbf{v}_1$. So for λ_1, there is even no need to solve Eq. (1.15), because we already know its solution. But what about the solution to $(\mathbf{A} - \lambda_2\mathbf{I})\mathbf{v} = 0$?

$$(\mathbf{A} - \lambda_2\mathbf{I})\mathbf{v} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 - v_2 \\ v_2 - v_1 \end{bmatrix} \tag{1.16}$$

So $(\mathbf{A} - \lambda_2\mathbf{I})\mathbf{v} = 0$ if and only if $v_1 = v_2$. Not surprisingly, we did not get a unique solution $\mathbf{v}$, because we found λ_2 precisely such that we get infinitely many solutions. It is not hard to see that vectors $\mathbf{v}$ for which $v_1 = v_2$ are all multiples of

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and constitute the SouthWest-NorthEast axis in the picture. Now the picture makes even more sense: the mapping $\mathbf{A}$ scales all the vectors along the NorthWest-SouthEast axis by 6 times, and all the vectors along the SouthWest-NorthEast axis by 4 times.

What about other vectors $\mathbf{x}$ that lie neither on the NorthWest-SouthEast axis nor on the SouthWest-NorthEast axis? Well, notice that $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent, and we can decompose any other vector into a linear combination of them (using standard orthogonal projection you learned in geometry):

$$\mathbf{x} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2$$

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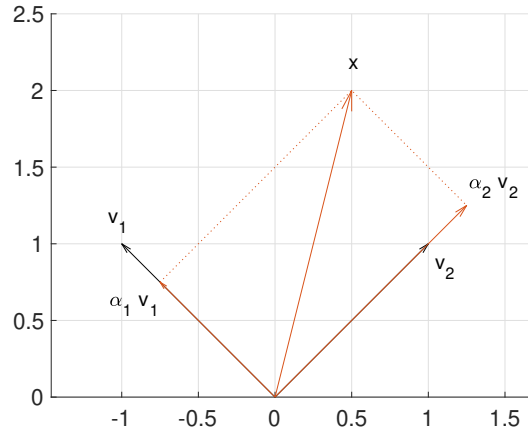
To find the unknown coefficients α_1 and α_2, you can simply use your knowledge of linear algebra!

$$\mathbf{x} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2$$

$$\mathbf{x} = \underbrace{\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}}_{\mathbf{V}} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$$

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \mathbf{V}^{-1} \mathbf{x}$$

Note that $\mathbf{V}$ is invertible precisely because $\mathbf{v}_1$ and $\mathbf{v}_2$ are independent. If they weren't, then we wouldn't be able to decompose $\mathbf{x}$ based on them. But fortunately, eigenvectors always turn out to be linearly independent (more on this later).

Now, we can clearly see what happens when $\mathbf{A}$ is applied to $\mathbf{x}$:

$$\mathbf{A}\mathbf{x} = \mathbf{A}(\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2)$$
$$= \alpha_1 \mathbf{A}\mathbf{v}_1 + \alpha_2 \mathbf{A}\mathbf{v}_2$$
$$= 6\alpha_1 \mathbf{v}_1 + 4\alpha_2 \mathbf{v}_2$$

So $\mathbf{A}$ scales the component of any vector along the NorthWest-SouthEast axis (along $\mathbf{v}_1$) by 6 times and its component along the SouthWest-NorthEast axis (along $\mathbf{v}_2$) by 4 times. Visually, this looks a bit like a combination of scaling (stretching) and rotation (rotation away from the $\pm\mathbf{v}_2$ and towards $\pm\mathbf{v}_1$), but there isn't any rotation really, it's just an unequal scaling.

Definition 1.5.2 (Eigenvalue and eigenvector) For any matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$, there exist precisely n numbers λ (potentially complex, and potentially repeated) for which the equation

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$$

has nonzero solutions. These numbers are called the eigenvalues of $\mathbf{A}$, and the corresponding nonzero vectors $\mathbf{v}$ that solve this equation are called the eigenvectors of $\mathbf{A}$. □

Let's see a few examples of finding eigenvalues and eigenvectors.

Example 1.5.3 (3x3 matrix with unique eigenvalues) Consider the matrix

$$\mathbf{A} = \begin{bmatrix} -5 & 3 & 7 \\ -5 & 3 & 5 \\ -4 & 2 & 6 \end{bmatrix}$$

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To find its eigenvalues, we need to solve the equation

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

$$\begin{vmatrix} -5-\lambda & 3 & 7 \\ -5 & 3-\lambda & 5 \\ -4 & 2 & 6-\lambda \end{vmatrix} = 0$$

$$-\lambda^3 + 4\lambda^2 - 6\lambda + 4 = 0$$

This a polynomial equation and we know, from Section 0.2, that it has exactly 3 (potentially repeated, potentially complex) roots. Using hand calculations or MATLAB, we can see that its solutions are

$$\lambda_1 = 2$$
$$\lambda_2 = 1 + j$$
$$\lambda_3 = 1 - j$$

which we sometimes show more compactly as

$$\lambda_1 = 2$$
$$\lambda_{2,3} = 1 \pm j$$

To find the eigenvectors associated with each eigenvalues, we simply solve the equation $(\mathbf{A} - \lambda_i\mathbf{I})\mathbf{v} = 0$, as we did in Eq. (1.16). Note that this is nothing but finding the null space of $\mathbf{A} - \lambda_i\mathbf{I}$. For λ_1, this becomes

$$(\mathbf{A} - 2\mathbf{I})\underbrace{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}_{\mathbf{v}} = 0 \Leftrightarrow \begin{cases} -7a + 3b + 7c = 0 \\ -5a + b + 5c = 0 \Leftrightarrow b = 5a - 5c \\ -4a + 2b + 4c = 0 \end{cases}$$

Substituting $b = 5a - 5c$ into the other two equations then gives

$$\begin{cases} 8a - 8c = 0 \\ 6a - 6c = 0 \end{cases} \Leftrightarrow c = a$$

Therefore, any vector

$$\mathbf{v}_1 = \begin{bmatrix} a \\ 0 \\ a \end{bmatrix}, \qquad \text{for any } a \in \mathbb{C}, a \neq 0$$

is an eigenvector corresponding to $\lambda_1 = 2$. This gives an entire line in the 3D space that is scaled by 2 by $\mathbf{A}$ (similar to the NorthWest-SouthEast and SouthWest-NorthEast directions for the matrix $\mathbf{A}$ in Eq. (1.14)). If you prefer (or are asked to provide) a single eigenvector associated with λ_1, pick any $a \in \mathbb{C}$ that you like, for example $\mathbf{v}_1 = \begin{bmatrix} 2j \\ 0 \\ 2j \end{bmatrix}$.

To find the eigenvector associated with λ_2, we proceed similarly:

$$(\mathbf{A} - (1+j)\mathbf{I})\begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0 \Leftrightarrow \begin{cases} -(6+j)a + 3b + 7c = 0 \\ -5a + (2-j)b + 5c = 0 \\ -4a + 2b + (5-j)c = 0 \Leftrightarrow b = 2a - \frac{5-j}{2}c \end{cases}$$

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Substituting the last equation into the first two gives

$$\begin{cases} -ja - (0.5 - 1.5j)c = 0 \\ -(1 + 2j)a + (0.5 + 3.5j)c = 0 \end{cases}$$

These equations may not immediately look multiples of each other, but they are in fact. To see, solve one and replace in the other:

$$\text{1st eq} \iff a = (1.5 + 0.5j)c \overset{\text{2nd eq}}{\Rightarrow} -(0.5 + 3.5j)c + (0.5 + 3.5j)c = 0$$

which holds for any c. Substituting $a = (1.5 + 0.5j)c$ into $b = 2a - \frac{5-j}{2}c$ also gives us $b = (0.5 + 1.5j)c$. Therefore, any vector

$$\mathbf{v}_2 = \begin{bmatrix} (1.5 + 0.5j)c \\ (0.5 + 1.5j)c \\ c \end{bmatrix}, \qquad \text{for any } c \in \mathbb{C}, c \neq 0$$

is an eigenvector associated with $\lambda_2 = 1 + j$. Again, if you want a single matrix, pick your choice of $c \neq 0$, such as $c = 2 \Rightarrow \mathbf{v}_2 = \begin{bmatrix} 3 + j & 1 + 3j & 2 \end{bmatrix}^T$.

Finally, to find the eigenvector associated with $\lambda_3 = 1 - j$, you can repeat the same process as above, which gives,

$$\mathbf{v}_3 = \begin{bmatrix} (1.5 - 0.5j)c \\ (0.5 - 1.5j)c \\ c \end{bmatrix}, \qquad \text{for any } c \in \mathbb{C}, c \neq 0$$

$\square$

Notice that $\mathbf{v}_3$ is the complex conjugate of $\mathbf{v}_2$. This is not by chance. <u>Whenever</u> you have two eigenvalues that are complex conjugates of each other, their corresponding eigenvectors are also complex conjugate of each other (can you prove this?).

In the examples that we have seen so far, all the eigenvalues of $\mathbf{A}$ have been distinct. This is not always the case. The following is an example.

Example 1.5.4 (3x3 matrix with repeated eigenvalues) This time consider the matrix

$$\mathbf{A} = \begin{bmatrix} -5 & 2 & -6 \\ 6 & -1 & 6 \\ 6 & -2 & 7 \end{bmatrix}$$

Similar to previous examples, eigenvalue are found using

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

$$\begin{vmatrix} -5 - \lambda & 2 & -6 \\ 6 & -1 - \lambda & 6 \\ 6 & -2 & 7 - \lambda \end{vmatrix} = 0$$

$$-\lambda^3 + \lambda^2 + \lambda - 1 = 0$$

This equation has three roots equal to

$$\lambda_1 = -1$$
$$\lambda_2 = \lambda_3 = 1$$

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two of which are repeated. This is perfectly fine. However, finding the eigenvectors becomes a bit more complicated. For $\lambda_1 = -1$ which is *not* repeated, everything is as before. You solve $(\mathbf{A} + \mathbf{I})\mathbf{v} = 0$ and will find that any vector

$$\mathbf{v}_1 = \begin{bmatrix} a \\ -a \\ -a \end{bmatrix}, \qquad \text{for any } a \in \mathbb{C}, a \neq 0$$

is an eigenvector corresponding to λ_1.

In order to find the eigenvectors corresponding to both of λ_2 and λ_3, we have to solve the same equation

$$(\mathbf{A} - \mathbf{I}) \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0 \Leftrightarrow \begin{cases} -6a + 2b - 6c = 0 \\ -6a + 2b - 6c = 0 \\ -6a + 2b - 6c = 0 \end{cases} \Leftrightarrow b = 3a + 3c$$

You can see that this time, we get only one equation in 3 variables, and we have two free variables to choose arbitrarily (I chose a and c, but any two would work). Therefore, any vector

$$\mathbf{v}_{2,3} = \begin{bmatrix} a \\ 3a + 3c \\ c \end{bmatrix} = a \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}, \qquad \text{for any } a, c \in \mathbb{C}, (a, c) \neq 0 \tag{1.17}$$

is an eigenvector corresponding to $\lambda_{2,3} = 1$. Notice the difference with the example before where the eigenvalues where different. When the eigenvalues where different, we found one <u>line</u> of eigenvectors corresponding to each eigenvalue. Now that we have a repeated eigenvalue, we found a <u>plane</u> of eigenvectors, with the same dimension (2) as the multiplicity of the repeated eigenvalue.

Similar to before, if you want (or are asked to) give <u>two specific eigenvectors</u> corresponding to λ_2 and λ_3 (instead of the plane of eigenvectors in Eq. (1.17)), you can pick any two <u>linearly independent</u> vectors from that plane, for example,

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$$

$\square$

The case of repeated eigenvalues can get more complex than this. In the above example, we were able to find two linearly independent vectors that satisfy $(\mathbf{A} - \mathbf{I})\mathbf{v} = 0$. In other words, the dimension of the null space of $\mathbf{A} - \mathbf{I}$ was 2, equal to the multiplicity of the repeated eigenvalues. We may not always be so lucky! Consider for example the matrix

$$\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$$

It is not hard to see that the two eigenvalues are $\lambda_1 = \lambda_2 = 3$. So we ideally would need two linearly independent vectors that satisfy $(\mathbf{A} - 3\mathbf{I})\mathbf{v} = 0$. This is impossible, because the null space of $\mathbf{A} - 3\mathbf{I}$ is only 1 dimensional. To see this, notice that

$$(\mathbf{A} - 3\mathbf{I}) \begin{bmatrix} a \\ b \end{bmatrix} = 0 \Leftrightarrow \begin{cases} b = 0 \\ 0 = 0 \end{cases}$$

So we can only choose a freely, but b must be zero, and only the vectors

$$\mathbf{v}_1 = \begin{bmatrix} a \\ 0 \end{bmatrix}, \qquad \text{for any } a \in \mathbb{C}, a \neq 0$$

are an eigenvector corresponding to $\lambda_1 = \lambda_2 = 3$. What about the other eigenvector $\mathbf{v}_2$? It doesn't exist! For these kinds of matrices where not enough eigenvectors can be found (which can *only* happen if we have repeated eigenvalues), we have to supplement the eigenvectors with additional vectors called "generalized eigenvectors". Good news, that's beyond our course!

Let's go back to Example 1.5.4 where we were lucky and able to find two linearly independent eigenvectors corresponding to the repeated eigenvalue. You might have noticed that not only $\mathbf{v}_2$ and $\mathbf{v}_3$ are linearly independent, but all three eigenvalues $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are linearly independent. The same was true in Example 1.5.3 where the eigenvalues were distinct. This is again not by chance:

Theorem 1.5.5 (Independence of eigenvectors) Consider a matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ that has distinct eigenvalues, or a matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ that has repeated eigenvalues but we are able to find as many linearly independent eigenvectors as the number of repeated eigenvalues. In both cases, the set of all eigenvectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is linearly independent. □

In the above theorem, I focused on a specific set of (square) matrices: those that either have distinct eigenvalues, or even if they have repeated eigenvalues, we are able to find as many independent eigenvectors as the multiplicity of each repeated eigenvalue. For reasons that we will see shortly, these matrices are called diagonalizable.

Before closing this section, here is how to do all of these in MATLAB.

MATLAB 1.5.6 (Eigenvalues & eigenvectors) The function `eig()` gives you the eigenvalues and eigenvectors. If you only want the eigenvalues, type

```
1   lambda = eig(A);
```

and it will give you a column vector `lambda` containing the eigenvalues of `A`. If you want the eigenvectors as well, type

```
1   [V, D] = eig(A);
```

and it will give you two matrices the same size as `A`. `D` is a diagonal matrix with the eigenvalues of `A` on its diagonal, and `V` is a matrix with eigenvectors of `A` as its columns (with the correct ordering, such that the i'th column of `V` is the eigenvector corresponding to the i'th diagonal element of `D`. □

1.6 Diagonalization

The process of diagonalization is one in which a square matrix is "transformed" into a diagonal one using a change of coordinates.

Consider a *diagonalizable* matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ (which you know what it means from the last section, right?). As always, we look at $\mathbf{A}$ as a map from $\mathbb{R}^n$ to $\mathbb{R}^n$:

$$\mathbf{y} = \mathbf{A}\mathbf{x} \tag{1.18}$$

Now let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of $\mathbf{A}$ (potentially repeated) and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be the corresponding linearly independent eigenvectors. The matrix

$$\mathbf{V} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix}$$

is nonsingular. So let's define

$$\hat{\mathbf{x}} = \mathbf{V}^{-1}\mathbf{x}, \qquad \hat{\mathbf{y}} = \mathbf{V}^{-1}\mathbf{y}$$

These vectors are the new coordinates of $\mathbf{x}$ and $\mathbf{y}$ in a coordinate system consisting of the columns of $\mathbf{V}$. But anyhow, substituting these into Eq. (1.18), we get

$$\mathbf{V}\hat{\mathbf{y}} = \mathbf{A}\mathbf{V}\hat{\mathbf{x}}$$
$$\hat{\mathbf{y}} = \underbrace{\mathbf{V}^{-1}\mathbf{A}\mathbf{V}}_{\hat{\mathbf{A}}}\hat{\mathbf{x}}$$
$$\hat{\mathbf{y}} = \hat{\mathbf{A}}\hat{\mathbf{x}}$$

Let's look at $\hat{\mathbf{A}} = \mathbf{V}^{-1}\mathbf{A}\mathbf{V}$ more closely. First, let us look at the product

$$\mathbf{A}\mathbf{V} = \mathbf{A}\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix}$$

Recall from Eq. (1.1) that this is equal to

$$\mathbf{A}\mathbf{V} = \begin{bmatrix} \mathbf{A}\mathbf{v}_1 & \mathbf{A}\mathbf{v}_2 & \cdots & \mathbf{A}\mathbf{v}_n \end{bmatrix}$$

but $\mathbf{v}_i$ are not any vectors, they are the eigenvectors of $\mathbf{A}$, so this simplifies to

$$\mathbf{A}\mathbf{V} = \begin{bmatrix} \lambda_1\mathbf{v}_1 & \lambda_2\mathbf{v}_2 & \cdots & \lambda_n\mathbf{v}_n \end{bmatrix}$$
$$= \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

Can you convince yourself of the last equality? Notice that this is nothing but the product of two matrices, so you can first break it into n matrix-vector products from Eq. (1.1), each of which is a linear combination from Eq. (1.2). The matrix

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

is an important matrix for us, because it contains the eigenvalues of $\mathbf{A}$ on its diagonal. Using this new notation, we get

$$\mathbf{A}\mathbf{V} = \mathbf{V}\mathbf{\Lambda}$$

or, in other words,

$$\hat{\mathbf{A}} = \mathbf{\Lambda}$$

This is called "diagonalization", and $\mathbf{\Lambda}$ is also called the diagonalization/diagonalized version of $\mathbf{A}$.

If these seem like a lot to digest, notice that this is what we did at the beginning of Section 1.5 where we were analyzing what the matrix $\begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix}$ does to vectors in the plane. After going step by step through the extraction of eigenvalues and eigenvectors, at the end we decomposed any vector $\mathbf{x}$ into its components along $\mathbf{v}_1$ and $\mathbf{v}_2$ and used the fact that in this new "coordinate system", $\mathbf{A}$ becomes a pure scaling in each direction, which is precisely what the diagonal matrix $\mathbf{\Lambda}$ does. Diagonalization will be a very valuable tool later in the study of linear systems.

Exercise 1.6.1 (Diagonalization) Diagonalize these matrices (find the eigenvalues and eigenvectors and check $\hat{\mathbf{A}} = \mathbf{V}^{-1}\mathbf{A}\mathbf{V}$ against $\boldsymbol{\Lambda}$):

- $\mathbf{A} = \begin{bmatrix} -4 & 1 & -2 \\ 0 & -4 & 0 \\ 1 & 4 & -2 \end{bmatrix}$

- $\mathbf{A} = \begin{bmatrix} -1 & 2 & 1 \\ 2 & -4 & -2 \\ 1 & -2 & -1 \end{bmatrix}$

□